

Growing Ising-like chain as a model of online emotional interactions

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We introduce a one-dimensional model basing on a special kind of asymmetrical one-side Ising-like dynamics. The model is equipped with the growth feature by adding a new node to the chain in each time-step. A spin of every succeeding node is influenced by the previous one but not vice-versa (the system is not hamiltonian). Owing to model's simplicity (it is an equivalent to a Markov chain) we are able exactly solve it in the presence of external field h . We conceive it as suitable for modeling for emotional online discussions arranged in a chronological order, where spin in every node conveys emotional valence of a subsequent post. We study the dependence of the chain's average spin $\langle s \rangle$ on external influence h (community tendency toward a selected valence) and noise level T (uncertainty of the emotional behavior). This leads us to an observation of three distinct phases - the first, where discussion evolution is determined by its launching message only, the second, in which the mostly observed emotion is coping the external influence, a finally, the third one, where the outcome is subject to fluctuations.

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Due to its simplicity and fully analytical treatment, one-dimensional models serve physicists as a very useful and comprehensible tool. Of the exceptional importance backed by feasibility of calculations is the Ising model [1]. Its basic version that considered only nearest neighbors coupling has been recursively modified to include, for example, long-range interactions $J_{ij} \sim |i - j|^{-\alpha}$ (i.e., no phase transition for $\alpha > 2$ [2] versus the existence of phase transition for $1 < \alpha \leq 2$ [3-5]). One-dimensional Ising model has also been subject to analysis with respect to a random field [6, 7], where it has been proved that the ratio of up spins to down spins exhibits devil's staircase [8]. Other research touches the issue of domain statistics [9], cluster size distribution [10] or inhomogeneity in the external field [11, 12]. In this paper we introduce a one-dimensional model basing on a asymmetrical one-side dynamics. However, the asymmetry is unlike the one proposed by Huang [13, 14], where the spin variable can take on two eigenvalues $+1$ and $-1/\lambda$ with $\lambda > 1$ nor it is connected to degeneration of higher-energy spin state [15]. Instead, we modify explicitly Ising hamiltonian by taking into account only left-hand neighbor. Moreover we also equip our model with growing (or evolving) component, i.e., we grow the chain by adding one node in each time-step.

Although extremely useful, one-dimensional models often suffer from exaggerated simplicity, i.g., one finds no evidence to support the idea that the agents related to social dynamics are to be distributed on a chain. In fact, in most cases, the relations between them tend to be highly embedded in a complex network space [16] — reflecting the actual topology of human contacts (private or professional). In this Letter we give clear reasons for choosing this very topology. Motivated by recent research [17, 18] we model the process of postings of emotional messages [19, 20] in a Internet forum. It is a common situation that the participants of such media use a chronological

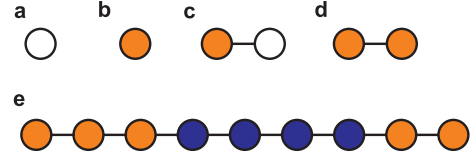


FIG. 1: (Color online). A scheme of the simulation process. (a-d) Consecutive steps of an exemplary simulation: (a) starting from an empty node, (b) adding random spin to the first node, (c) adding the next node, (d) inserting spin according to dynamics rule. (e) An effect of the simulation. Orange (gray, $s = +1$) and blue (dark gray, $s = -1$) discs symbolize spins.

structure of the incoming posts that can be easily regarded as a one-dimensional chain (i.e., the consecutive posts are represented by the nodes in the chain). The results of our previous analysis [17, 18] indicate that one of the most dominant phenomena seen in the online discussions is the emotional homophily. It follows that the discussion's participants take into account the history of the thread and basing on that knowledge issue their own posts, trying to imitate the overall emotional tendency. The simplest possible way of implementing this behavior is to take into account only the last comment (i.e., the newest one).

Model description. The model bases on the idea of a growing chain (see Fig. 1). The process is organized as follows: in the first step, a random variable $s_0 = \pm 1$, called spin (an equivalent of emotional valence [21] in online discussion), drawn with probability $\Pr(s_0 = \pm 1) = 1/2$ is inserted into the first node of the chain (Fig. 1a-b). Then, another node of the chain is added to the right of the last one (Fig. 1c) and it is initially equipped spin once again drawn with equal probabilities $\Pr(s_1 = \pm 1) = 1/2$. Subsequently the node becomes a subject to the updating procedure that is bas-

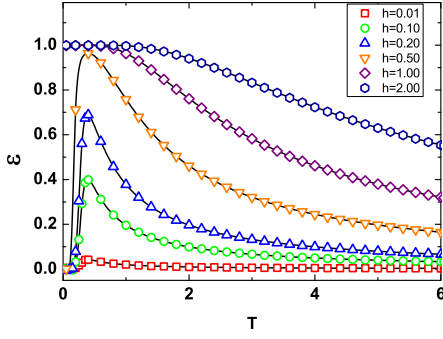


FIG. 2: (Color online) Average spin $\langle s \rangle$ as function of temperature T for different values of the external magnetic field $h = 0.01$ (squares), $h = 0.1$ (circles), $h = 0.2$ (upward triangles), $h = 0.5$ (downward triangles), $h = 1.0$ (diamonds), $h = 2.0$ (hexagons). Solid lines come from Eqs (6) and (7). All data points are for $N = 10^3$ and averaged over $M = 10^5$ simulations.

ing on the Ising-like model approach. For each new node i we define $\mathcal{E}_i = -J e_{i-1} e_i - h e_i$, where J is an equivalent of exchange energy and h is the external field. The function \mathcal{E}_i promotes spins of the same sign in the consecutive nodes of the chain and can be treated as an *emotional discomfort* function. As the spin is drawn, we test how flipping its sign to the opposite one (i.e., from $s_i = +1$ to $s_i = -1$ or likewise) affects the change of function \mathcal{E} as $\Delta \mathcal{E} = \mathcal{E}'_i - \mathcal{E}_i = -(J s_{i-1} + h)(s'_i - s_i)$, where term \mathcal{E}'_i corresponds to s_i calculated when $s_i \rightarrow s'_i = -s_i$. Then we follow the Metropolis algorithm [22] i.e., if the $\Delta \mathcal{E} < 0$ we accept the change, otherwise we test if the expression $\exp(-\beta \Delta \mathcal{E})$ is smaller or larger than a random value $\xi \in [0; 1]$ (here $\beta = \frac{1}{k_B T}$, where k_B is Boltzmann constant and T is the temperature). If the latter occurs we accept the change otherwise the spin is kept as originally chosen. The procedure of adding new nodes and setting their spin variables according to the above described rules is repeated until size N of the chain is reached (Fig. 1e). As an outcome, the average spin in the chain (an equivalent of the average emotion in the discussion), defined as $\langle s \rangle = \frac{1}{N} \sum_{i=1}^N s_i$ is calculated. It is then averaged over M realizations (typically, in this study $M = 10^5$).

Numerical simulations.— All numerical simulations have been performed for $J = k_B = 1$. Figure 2 shows the average spin $\langle s \rangle$ as a function of the temperature T for selected values of external field h . In the case of $h < 1$ the plot reveals $\langle s \rangle$ equal to zero for small $T \ll 1$, then a clear maximum for some specific value T_c appears. Finally, a recoil toward zero for $T > T_c$ takes place. In the case of $h \geq 1$ such a phenomenon is not observed: instead, $\langle s \rangle = 1$ for small T and then there is a decrease toward zero.

The explanation of such behavior is following: if the field h is smaller than the value $J = 1$ of influence from the previous post and the noise level (temperature) T is low the $\langle s \rangle$ follows the spin value of the first node of the

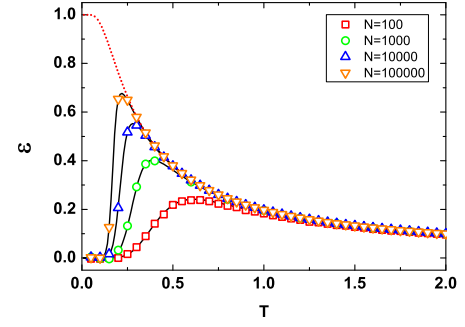


FIG. 3: (Color online) Average spin $\langle s \rangle$ as function of temperature T for different values of the chain size $N = 10^2$ (squares), 10^3 (circles), 10^4 (upward triangles), 10^5 (downward triangles), 10^6 (diamonds). All data points for $h = 0.1$ and are averaged over $M = 10^5$ simulations. Solid lines come from Eq. (6) and the dotted line is $\tanh(2h/T)$.

chain. Let us note that in our system the initial condition for the node s_0 is also a system boundary condition. Taking into account the fact that there is an equal chance of starting from either $s_0 = +1$ or $s_0 = -1$ this leads to a situations where half of the realizations are $\langle s \rangle = +1$ and half $\langle s \rangle = -1$. While temperature increases the fluctuations cancel the effect of influence of the launching spin (memory about system initial conditions) and more and more spins tend to align according to the sign of the external field leading to the observed peak. Since fluctuations destroy also the ordering impact of the field h in the *whole system* thus after crossing some characteristic temperature T_c the average spin is lowered. Since the effect results from an interplay between initial conditions external field h and the fluctuations thus the temperature T_c should be small for larger systems when even small fluctuations destroy the initial condition influence (it will discussed later). If the field h is stronger than the influence from the launching node ($h > 1$) then every next spin follows this field thus thermal fluctuations T can only lower the mean spin $\langle s \rangle$. Such a behavior is in fact observed at two upper plots of Fig. 2.

Figure 3 shows that indeed for smaller systems (e.g., $N = 10^2$ the characteristic temperature T_c is of order 0.1 – 1 and is shifted toward lower values as compared to systems of size $N = 10^6$. An observed maximum $\langle s \rangle(T_c)$ is smaller for smaller systems since the influence of initial conditions plays for them a more important role.

It is also interesting to track the dependence of the average spin value on the level of the external field (see Fig. 4). In the case of low temperatures ($T < T_c$) average spin value changes abruptly from $\langle s \rangle = -1$ to $\langle s \rangle = 0$ for $h = -1$ and then from $\langle s \rangle = 0$ to $\langle s \rangle = 1$ for $h = 1$. For higher temperatures ($T > T_c$) this change is smoother and length range of h for which $\langle s \rangle = 0$ is smaller.

Analytical description.— For modeling purposes we start with checking the way the probability of positive emotion following another positive emotion is calculated in the presence of the external impact $h \geq 0$. First, we set

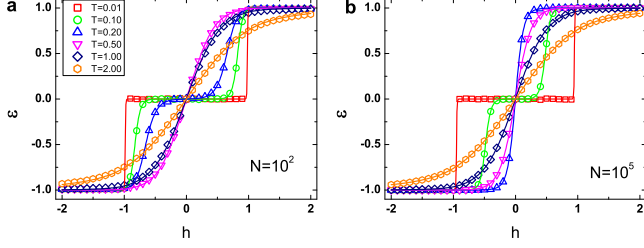


FIG. 4: (Color online) Average spin $\langle s \rangle$ as function of the external field h for different values of temperature $T = 0.01$ (squares), $T = 0.1$ (circles), $T = 0.2$ (upward triangles), $T = 0.5$ (downward triangles), $T = 1.0$ (diamonds), $T = 2.0$ (hexagons). Curves for (a) $N = 10^2$, (b) $N = 10^5$. All data points are averaged over $M = 10^5$ simulations. Solid lines come from Eq. (6).

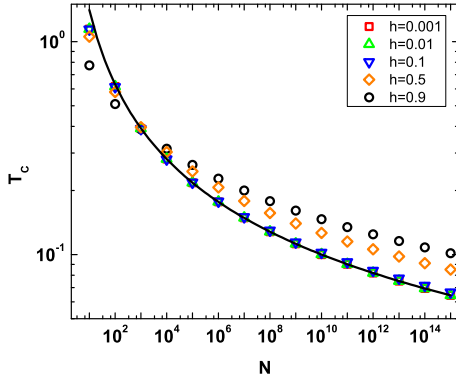


FIG. 5: (Color online) Characteristic temperature T_c versus the size of the chain N . Symbols (upward triangles for $h = 0.01$, downward triangles for $h = 0.1$, diamonds for $h = 0.5$ and circles for $h = 0.9$) are numerical solution of Eq. (6) while the solid line comes from Eq. (9).

$s_0 = 1$. Then, with equal probabilities $\Pr(s_1 \pm 1) = 1/2$, the spin in the next node is chosen to be positive or negative. Next, we calculate the change of function \mathcal{E} : (i) if $s_1 = +1$ and $s'_1 = -1$ then $\Delta\mathcal{E} = 2(h + J) > 0$, so the change is accepted with probability equal to $e^{-\tilde{\beta}(h+J)}$ and not accepted with probability equal to $1 - e^{-\tilde{\beta}(h+J)}$, where $\tilde{\beta} = 2/(k_B T)$, (ii) if $s_1 = -1$ and $s'_1 = +1$ then $\Delta\mathcal{E} = -2(h + J) < 0$, so the change is always accepted. As a consequence the probability p_{++} of positive spin following another positive spin is equal to $p_{++} = \frac{1}{2}(1 - e^{-\tilde{\beta}(h+J)}) + \frac{1}{2} \times 1 = 1 - \frac{1}{2}e^{-\tilde{\beta}(h+J)}$. Following the same line of thought it is possible to obtain the exact expressions for the probabilities $p_{++}, p_{+-}, p_{-+}, p_{--}$ depending on the range of h . Then, for $h \in [0; J)$ we have

$$\begin{cases} p_{++} = \Pr(+|+) = 1 - \frac{1}{2}e^{-\tilde{\beta}(h+J)} \\ p_{+-} = \Pr(-|+) = 1 - p_{++} \\ p_{--} = \Pr(-|-) = 1 - \frac{1}{2}e^{\tilde{\beta}(h-J)} \\ p_{-+} = \Pr(+|-) = 1 - p_{--} \end{cases} \quad (1)$$

while for $h \geq J$ we obtain

$$\begin{cases} p_{++} = \Pr(+|+) = 1 - \frac{1}{2}e^{-\tilde{\beta}(h+J)} \\ p_{+-} = \Pr(-|+) = 1 - p_{++} \\ p_{--} = \Pr(-|-) = \frac{1}{2}e^{-\tilde{\beta}(h-J)} \\ p_{-+} = \Pr(+|-) = 1 - p_{--} \end{cases} \quad (2)$$

For simplicity of the further calculations we use the following notation: we denote the probability to stay in +1 state as $p_{++} = p$, the probability to move from state +1 to state -1 is $p_{+-} = 1 - p$. Similarly we denote the probability to stay in -1 as $p_{--} = q$ and the probability to move from state -1 to +1 is $p_{-+} = 1 - q$. In effect we obtain the transition matrix \mathbf{P}

$$\mathbf{P} = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix} \quad (3)$$

that defines probabilities evolution of both states $\Pr(s_n = \pm 1)$ as $\mathbf{s}_{n+1} = \mathbf{s}_n \mathbf{P}$, where $\mathbf{s}_n = [\Pr(s_n = +1) \ \Pr(s_n = -1)]$. Thus the evolution of \mathbf{s}_n is in fact an equivalent of a two-state Markov chain [23] governed by the transition matrix \mathbf{P} that has a solution $\mathbf{s}_n = \mathbf{s}_0 \mathbf{P}^n$, where $\mathbf{s}_0 = [1/2 \ 1/2]$ and

$$\mathbf{P}^n = \begin{bmatrix} \frac{q-1+(p-1)(q+p-1)^n}{(q-1)[1-(q+p-1)^n]} & \frac{(p-1)[1-(q+p-1)^n]}{p-1+(q-1)(q+p-1)^n} \\ \frac{q+p-2}{q+p-2} & \frac{q+p-2}{q+p-2} \end{bmatrix} \quad (4)$$

This leads to calculation of the average spin in the n -th node as $\langle s_n \rangle = \mathbf{s}_n [1 \ -1]^T$. Finally, performing the mean of $\langle e_n \rangle$ over all nodes in the chain gives the average spin in the chain:

$$\langle s \rangle = \frac{p-q}{2-p-q} \left[1 + \frac{1}{N} - \frac{1-(p+q-1)^{N+1}}{N(2-p-q)} \right], \quad (5)$$

Similar calculations can be performed for ranges $h \in (-\infty; -J]$ and $h \in (-J; 0]$. The symmetry of the problem results in swapping all the indices "+" to "-" and likewise in Eqs (1-2). As an outcome we obtain a rotated matrix \mathbf{P} that leads again to Eq. (5). In effect by applying the of exact values of probabilities p and q given by Eqs (1-2) and obtain the relations that bind the average spin value with external field and temperature for ranges $|h| < J$:

$$\langle s \rangle_s = \tanh \tilde{\beta} h \left[1 + \frac{1}{N} - \frac{1 - (1 - e^{-\tilde{\beta} J} \cosh \tilde{\beta} h)^{N+1}}{N e^{-\tilde{\beta} J} \cosh \tilde{\beta} h} \right] \quad (6)$$

and $|h| \geq J$:

$$\langle s \rangle_l = \frac{\cosh \tilde{\beta} J - e^{\tilde{\beta} h}}{\sinh \tilde{\beta} J - e^{\tilde{\beta} h}} \left[1 + \frac{1}{N} - \frac{1 - (e^{-\tilde{\beta} h} \sinh \tilde{\beta} J)^{N+1}}{N (1 - e^{-\tilde{\beta} h} \sinh \tilde{\beta} J)} \right] \quad (7)$$

The above equations are subject to further approximation. For example, $\langle s \rangle_s \approx \tanh \tilde{\beta} h$ for large N (see dotted line in Fig. 3), while for large both N and T we have

$\langle s \rangle \approx \tilde{\beta}h$ for the whole range of h . Equations (6-7) fully support (solid lines in Figs 2 and 4) numerical observations. Both — numerical and analytical — approaches indicate crucial role played by the system's size N (see Fig. 5): for a constant value of external field h increasing N leads to a shift in T_c toward $T = 0$ as well as to increase of the value $\langle s \rangle(T_c)$. It is essential to consider its exact value; first of all, as it has been shown before in the previous Section this phenomenon is observed only for $h < 1$. Then the formal solution can be obtained from Eq. (6) but it is only possible to treat it numerically (see symbols in Fig. 5). To get an analytical estimation we assume that $\beta h \ll 1$ which gives the opportunity to rewrite Eq. (6) as

$$\langle s \rangle_s \approx \tilde{\beta}h \left[1 + \frac{1}{N} - \frac{1 - (1 - e^{-\tilde{\beta}J})^{N+1}}{Ne^{-\tilde{\beta}J}} \right] \quad (8)$$

The above expression is an equivalent of magnetic susceptibility $\left(\frac{\partial \langle s \rangle_s}{\partial h} \right)_{h=0}$ times h . If we further assume $N \gg 1$, and solve $\frac{\partial \langle s \rangle_s}{\partial T} = 0$ one can approximate T_c as

$$T_c \approx \frac{2J}{k_B [W(Ne) - 1]} \quad (9)$$

where $W(\dots)$ is Lambert W function. Comparison between this approximation and numerical solution of Eq. (6) is shown in Fig. 5, providing evidence of good agreement for small values of h as expected. It follows that for real-world examples where the discussion length probability follows a power-law function $p(N) \sim N^{-\gamma}$ and $N_{max} \approx 10^5$ [17] the effect of non-zero T_c could be observed.

Discussion. — It is of use to compare and contrast the obtained results with the classical one-dimensional Ising model (e.g., [24]). The most noticeable difference regards undoubtedly the foundations of the model — in the classical case the length of the chain is fixed and each node i is initially filled with a spin $s_i = \pm 1$. The thermalization procedure involves energy couplings with both neighbors of node i and it is performed repeatedly. Here, in the presented model, the chain grows by adding new nodes and each node is subject to the thermalization procedure only *once*, moreover just its left-hand neighbor is taken into account. The magnetization per spin (an equivalent of $\langle e \rangle$) in the case of the classical one-dimensional Ising

model is given by

$$m(h, T, N) = \frac{\sinh \beta h}{\sqrt{\sinh^2 \beta h + e^{-4\beta J}}} \frac{1 - \left(\frac{\lambda_-}{\lambda_+} \right)^N}{1 + \left(\frac{\lambda_-}{\lambda_+} \right)^N} \quad (10)$$

where $\lambda_{\pm} = e^{\beta J} \left(\cosh \beta h \pm \sqrt{e^{-4\beta J} + \sinh^2 \beta h} \right)$ are the eigenvalues of the transfer matrix [24]. In our model the size of the chain plays a pivotal role, governing the position and height of the maximum of \bar{e} for $h < J$. Although Eq. (9) proves that in the limit of $N \rightarrow \infty$ the value $T_c \rightarrow 0$, nonetheless the convergence is slow (approximately as $1/\ln N$). For the classical Ising model the term containing system's size rapidly converges to 1: e.g., for $J = k_B = 1$, $h = 0.1$ and $T = 0.1$ one needs as little as $N = 10$ to have the convergence. On the other hand, in our model a large chain length is needed. For small T the terms in square brackets in Eq. (6) can be approximated by $\frac{1}{2}(N+1)e^{-\tilde{\beta}J} \cosh \tilde{\beta}h$ which in turn can be used to estimate the critical value of N for which f is equal to one as $N_c \approx 4e^{\tilde{\beta}(J-h)}$. Then, for the same parameters ($J = k_B = 1$, $h = 0.1$, $T = 0.1$) $N_c = 2.62 \times 10^8$.

Finally, opposite to our model, the magnetization in the classical Ising model for $J > 0$ (ferromagnetic case) is a strictly monotonous decaying function of T starting from $m = 1$ for $h > 0$. The maximum is observed for $J < 0$ (antiferromagnetic case) but it is independent from N (in the sense of factor's f_l rapid convergence to 1) and for $h \ll 1$ we have $T_c = 2$.

Summary. — In this paper we proposed and exactly solved a one-dimensional model basing on a growing chain with an asymmetrical Ising-like dynamics. The model can be considered as a certain modification of the original Ising concept accomplished by restriction imposed on the hamiltonian function. Moreover, we also introduced growth factor into the model, expanding the chain with one node in each time-step. Those features enabled simple analytical treatment (as Markov chain) that produces exact expressions for an equivalent of magnetization in the classical Ising model. The model, thought as a construct resembling on-line discussions, combines three different elements (i) chain-like structure of on-line discussions, (ii) tendency to take into account only the previous comment, (iii) global interactions inserted in a form of external field and fluctuations.

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